Warsaw University of Technology

Faculty of Power and Aeronautical Engineering

WARSAW UNIVERSITY OF TECHNOLOGY

Institute of Aeronautics and Applied Mechanics

Finite element method (FEM)

Part 3. Two dimensional (2D) cases. CST

03.2021

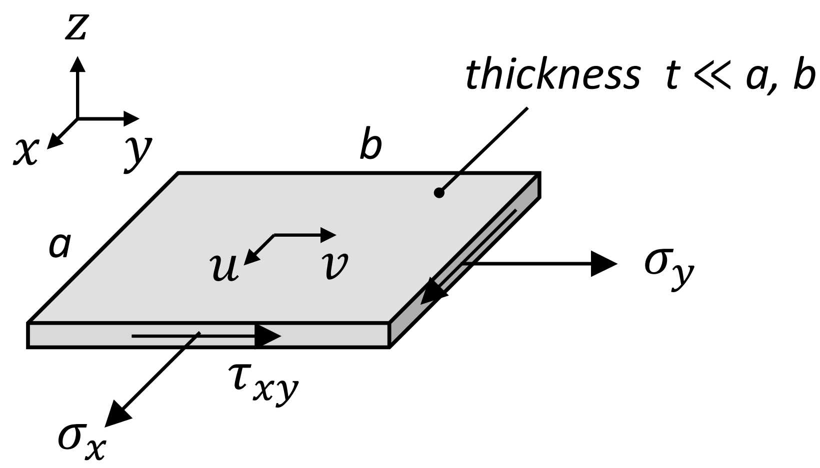
Plane stress (thin plates, shells)

$\sigma_x \quad ; \quad \sigma_y \quad ; \quad \sigma_z = 0$

$\tau_{xy} \quad ; \quad \tau_{yz} = 0 \quad ; \quad \tau_{zx} = 0$

$\varepsilon_x \quad ; \quad \varepsilon_y \quad ; \quad \varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$

$\gamma_{xy} \quad ; \quad \gamma_{yz} = 0 \quad ; \quad \gamma_{zx} = 0$



$$\underset{1\times 2}{\lfloor u \rfloor} = \lfloor u, v \rfloor$$

$$\underset{1\times 3}{\lfloor \sigma \rfloor} = \lfloor \sigma_x, \sigma_y, \tau_{xy} \rfloor$$

$$\underset{1\times 3}{\lfloor \varepsilon \rfloor} = \lfloor \varepsilon_x, \varepsilon_y, \gamma_{xy} \rfloor$$

$$\underset{\substack{3\times 3 \\ P.STRESS}}{[D]} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}$$

$$\underset{3\times 2}{[R]} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}$$

$$\underset{3\times 1}{\{\sigma\}} = \underset{3\times 3}{[D]} \underset{3\times 1}{\{\varepsilon\}}$$

$$\underset{3\times 1}{\{\varepsilon\}} = \underset{3\times 2}{[R]} \underset{2\times 1}{\{u\}}$$

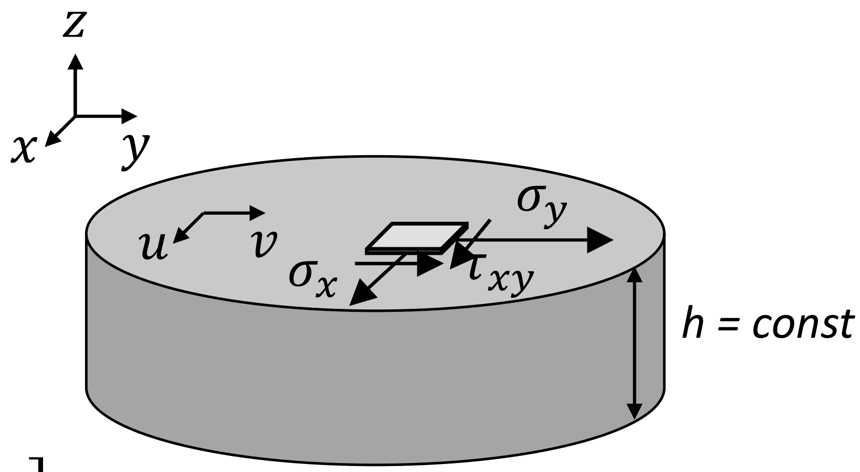
Plane strain (infinitely long pipe, prism and roller)

σ_x ; σ_y ; $\sigma_z = \nu(\sigma_x + \sigma_y)$

τ_{xy} ; $\tau_{yz} = 0$; $\tau_{zx} = 0$

ε_x ; ε_y ; $\varepsilon_z = 0$

γ_{xy} ; $\gamma_{yz} = 0$; $\gamma_{zx} = 0$



$$\underset{1\times 2}{\lfloor u \rfloor} = \lfloor u, v \rfloor$$

$$\underset{1\times 3}{\lfloor \sigma \rfloor} = \lfloor \sigma_x, \sigma_y, \tau_{xy} \rfloor$$

$$\underset{1\times 3}{\lfloor \varepsilon \rfloor} = \lfloor \varepsilon_x, \varepsilon_y, \gamma_{xy} \rfloor$$

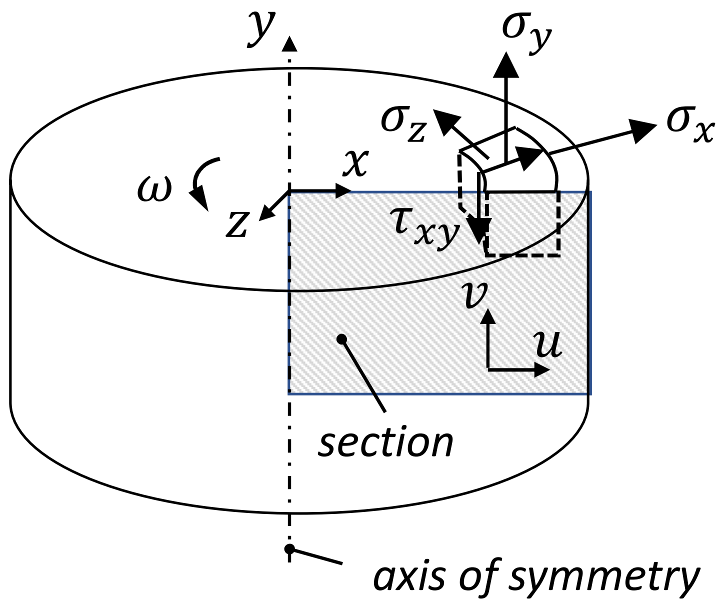
$$\underset{3\times 2}{[R]} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}$$

$$\underset{3\times 1}{\{\sigma\}} = \underset{3\times 3}{[D]}\,\underset{3\times 1}{\{\varepsilon\}}$$

$$\underset{3\times 1}{\{\varepsilon\}} = \underset{3\times 2}{[R]}\underset{2\times 1}{\{u\}}$$

$$\underset{P.STRAIN}{\underset{3\times 3}{[D]}} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix}$$

Axisymmetry (rotating disc)



σ_x ; σ_y ; σ_z

τ_{xy} ; $\tau_{yz} = 0$; $\tau_{zx} = 0$

ε_x ; ε_y ; $\varepsilon_z = 0$

γ_{xy} ; $\gamma_{yz} = 0$; $\gamma_{zx} = 0$

directions:
x – radial
y – longitudinal
z – hoop

$$\underset{1\times 2}{\lfloor u \rfloor} = \lfloor u, v \rfloor$$

$$\underset{1\times 4}{\lfloor \sigma \rfloor} = \lfloor \sigma_x, \sigma_y, \sigma_z, \tau_{xy} \rfloor$$

$$\underset{1\times 4}{\lfloor \varepsilon \rfloor} = \lfloor \varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy} \rfloor$$

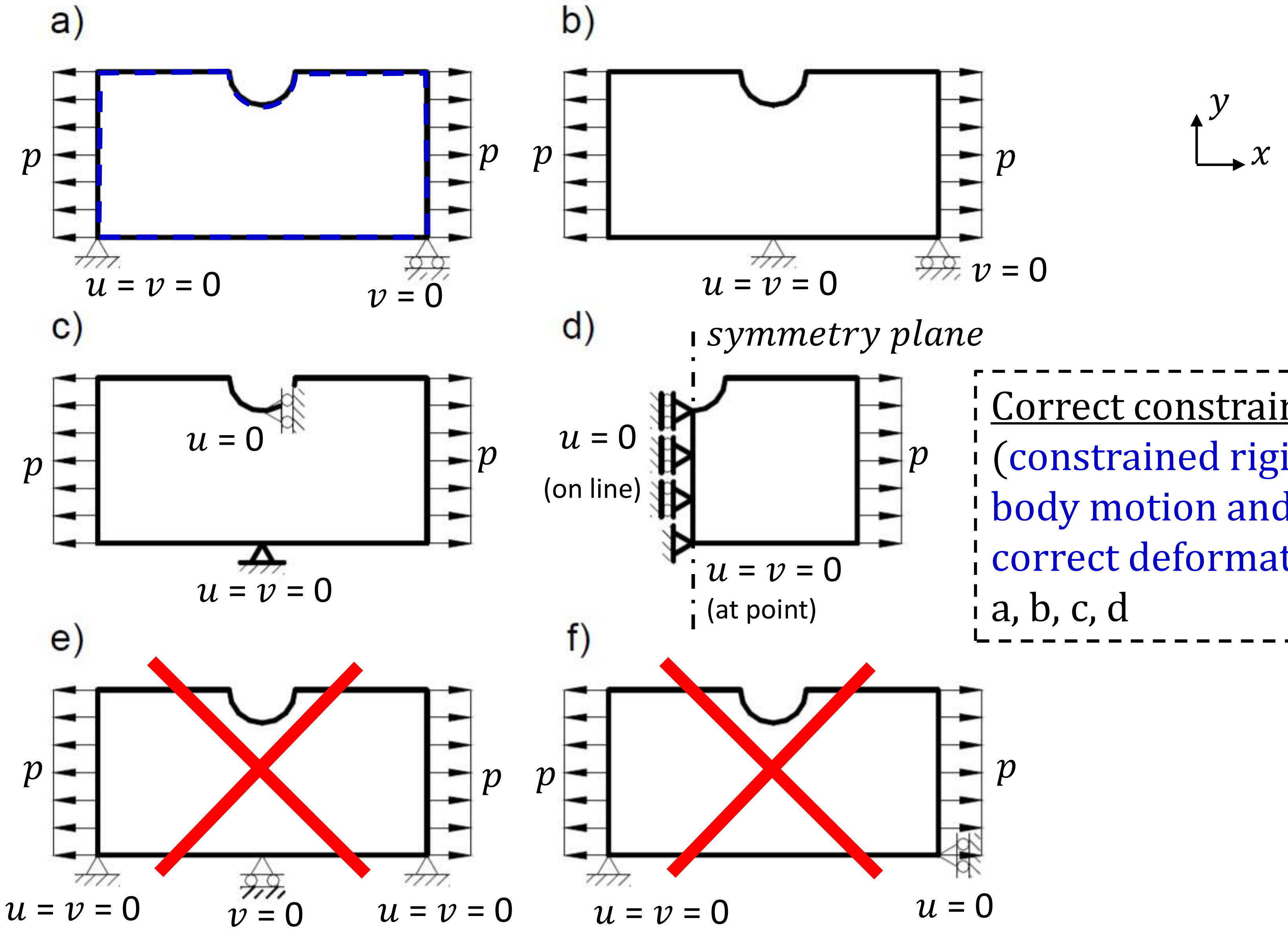
$$\underset{4\times 2}{[R]} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ \dfrac{1}{x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}$$

$$\underset{4\times 1}{\{\sigma\}} = \underset{4\times 4}{[D]}\,\underset{4\times 1}{\{\varepsilon\}}$$

$$\underset{4\times 1}{\{\varepsilon\}} = \underset{4\times 2}{[R]}\underset{2\times 1}{\{u\}}$$

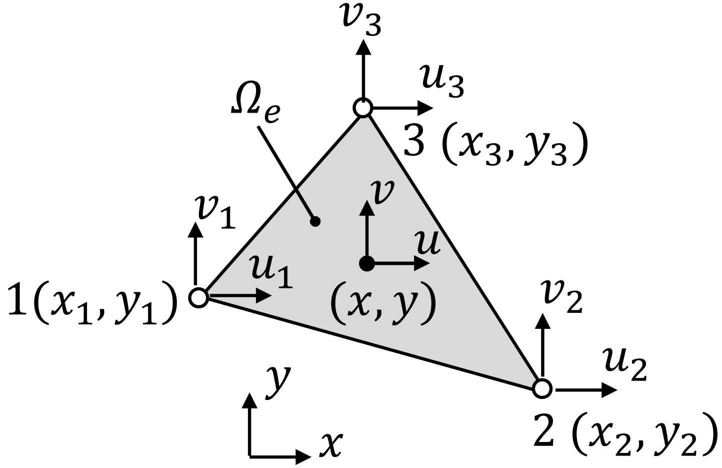
$$\underset{\substack{4\times 4 \\ AXISYMMETRY}}{[D]} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & 0.5-\nu \end{bmatrix}$$

Constraints for a 2D plate loaded by forces being in equilibrium

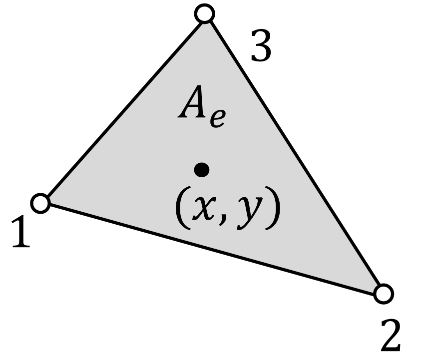


Correct constraints:
(constrained rigid body motion and correct deformation):
a, b, c, d

CST finite element (2D, 3-node triangle)

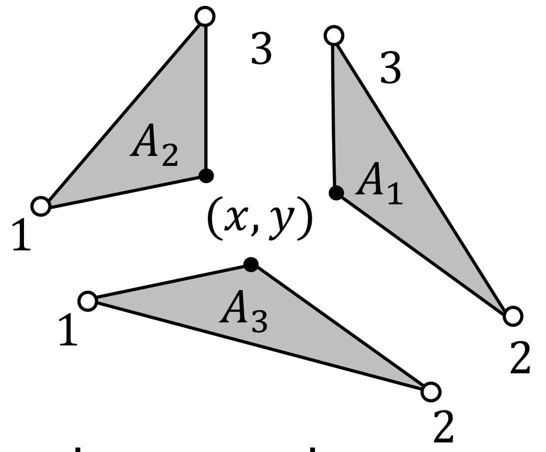


$$\int_{\Omega_e} d\Omega_e = \underbrace{A_e}_{\text{area}} \cdot \underbrace{t_e}_{\text{thickness}}$$



$$A_e = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \frac{x_2 y_3 - x_3 y_2 + x_3 y_1 - x_1 y_3 + x_1 y_2 - x_2 y_1}{2}$$

$n = 3 \quad ; \quad n_p = 2 \quad \rightarrow \quad n_e = n \cdot n_p = 6$



Area coordinates as functions of coordinates (x, y):

$$A_e = A_1(x, y) + A_2(x, y) + A_3(x, y)$$

$$A_1 = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x & x_2 & x_3 \\ y & y_2 & y_3 \end{vmatrix} \quad ; \quad A_2(x, y) = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x & x_3 \\ y_1 & y & y_3 \end{vmatrix} \quad ; \quad A_3(x, y) = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x \\ y_1 & y_2 & y \end{vmatrix}$$

Shape functions of the CST element

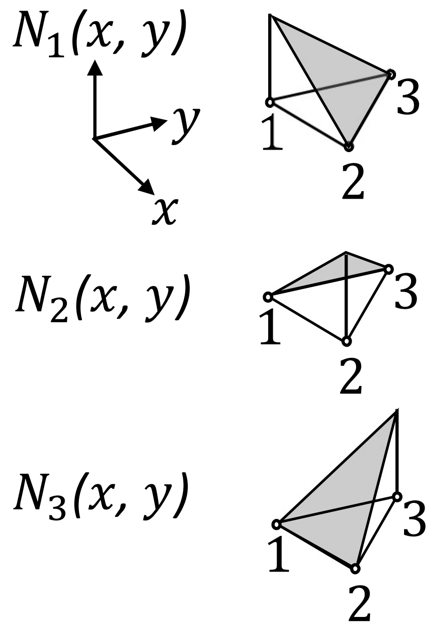
shape functions = normalized area coordinates:

$$N_1(x,y) = \frac{A_1(x,y)}{A_e} = \frac{1}{2A_e}(a_1 + b_1 x + c_1 y)$$

$$N_2(x,y) = \frac{A_2(x,y)}{A_e} = \frac{1}{2A_e}(a_2 + b_2 x + c_2 y)$$

$$N_3(x,y) = \frac{A_3(x,y)}{A_e} = \frac{1}{2A_e}(a_3 + b_3 x + c_3 y)$$

$$N_1(x,y) + N_2(x,y) + N_3(x,y) = 1$$



node	$N_1(x,y)$	$N_2(x,y)$	$N_3(x,y)$
1	1	0	0
2	0	1	0
3	0	0	1

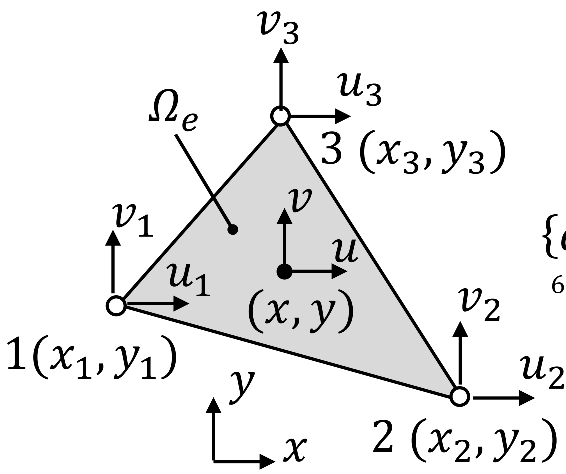
where:

$a_1 = x_2 y_3 - x_3 y_2$; $a_2 = x_3 y_1 - x_1 y_3$; $a_3 = x_1 y_2 - x_2 y_1$

$b_1 = y_2 - y_3$; $b_2 = y_3 - y_1$; $b_3 = y_1 - y_2$

$c_1 = x_3 - x_2$; $c_2 = x_1 - x_3$; $c_3 = x_2 - x_1$

Isoparametric mapping in the CST element



$$\underset{6\times 1}{\{q\}_e} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}_e$$

vector of shape functions:

$$\underset{1\times 3}{\lfloor N(x, y) \rfloor} = \lfloor N_1(x, y), N_2(x, y), N_3(x, y) \rfloor$$

vectors of nodal coordinates;

$$\underset{3\times 1}{\{x_i\}_e} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \quad ; \quad \underset{3\times 1}{\{y_i\}_e} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix}$$

coordinates of any point are based on shape functions and nodal parameters:

$$x = \underset{1\times 3}{\lfloor N(x, y) \rfloor} \underset{3\times 1}{\{x_i\}_e} = N_1(x, y) x_1 + N_2(x, y) x_2 + N_3(x, y) x_3$$

$$y = \underset{1\times 3}{\lfloor N(x, y) \rfloor} \underset{3\times 1}{\{y_i\}_e} = N_1(x, y) y_1 + N_2(x, y) y_2 + N_3(x, y) y_3$$

displacements at any point:

$$\underset{2\times 1}{\{u(x, y)\}} = \underset{2\times 6}{[N(x, y)]} \underset{6\ \times 1}{\{q\}_e}$$

Isoparametric mapping- the same shape functions used for geometry and displacements

Strain-displacement matrix of the CST element

strain vector for plane stress or plane strain conditions:

$$\underset{3\times 1}{\{\varepsilon\}} = \underset{3\times 2}{[R]}\ \underset{2\times 1}{\{u\}} = \underset{3\times 2}{[R]}\ \underset{2\times 6}{[N]}\underset{6\times 1}{\{q\}_e} =$$

$$= \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} N_1(x,y) & 0 & N_2(x,y) & 0 & N_3(x,y) & 0 \\ 0 & N_1(x,y) & 0 & N_2(x,y) & 0 & N_3(x,y) \end{bmatrix} \underset{6\times 1}{\{q\}_e} =$$

$$= \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix} \underset{6\times 1}{\{q\}_e} = \underset{3\times 6}{[B]}\ \underset{6\times 1}{\{q\}_e}$$

$$[B] = \frac{1}{2A_e} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix} \rightarrow \underset{3\times 1}{\{\varepsilon\}} = \underset{3\times 6}{[B]}\ \underset{6\times 1}{\{q\}_e}$$ – strain is constant

$$\underset{3\times 1}{\{\sigma\}} = \underset{3\times 3}{[D]}\ \underset{3\times 1}{\{\varepsilon\}}$$ – stress is constant

CST – Constant Strain Triangle

Elastic strain energy in the CST element. Local stiffness matrix

elastic strain energy in a finite element:

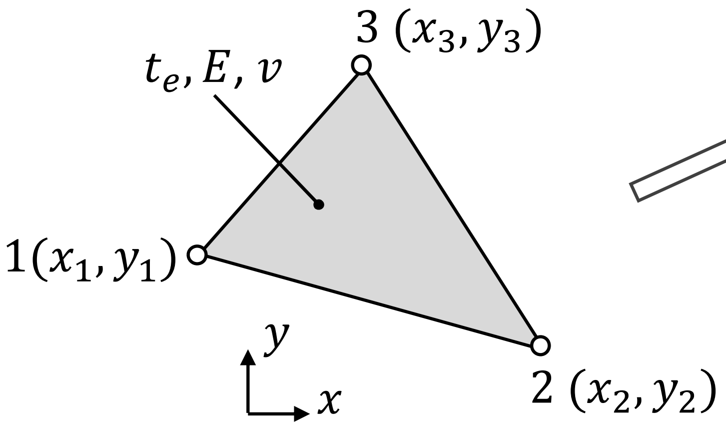
$$U_e = \frac{1}{2}\int_{\Omega_e} \underset{1\times3}{\lfloor\varepsilon\rfloor}\underset{3\times1}{\{\sigma\}}\, d\Omega_e = \frac{1}{2}\underset{1\times3}{\lfloor\varepsilon\rfloor}\underset{3\times1}{\{\sigma\}}\int_{\Omega_e} d\Omega_e = \frac{1}{2}\underset{1\times6}{\lfloor q\rfloor_e}\underset{6\times3}{[B]^T}\underset{3\times3}{[D]}\underset{3\times6}{[B]}\underset{6\times1}{\{q\}_e}\, A_e t_e =$$

$$\underset{3\times1}{\{\sigma\}} = \underset{3\times3}{[D]}\ \underset{3\times1}{\{\varepsilon\}}$$

$$= \frac{1}{2}\lfloor q\rfloor_e [k]_e \{q\}_e$$

$$\underset{1\times3}{\lfloor\varepsilon\rfloor} = \underset{1\times6}{\lfloor q\rfloor_e}\underset{6\times3}{[B]^T} \qquad \underset{3\times1}{\{\varepsilon\}} = \underset{3\times6}{[B]}\underset{6\times1}{\{q\}_e}$$

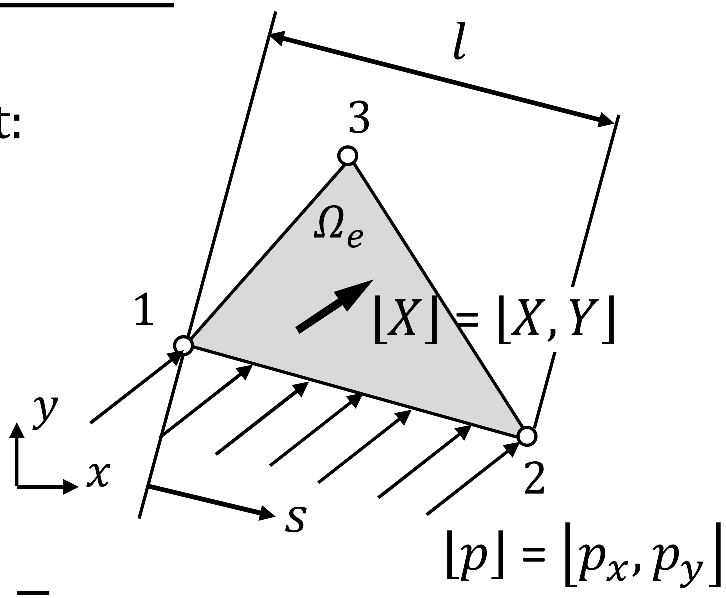
local stiffness matrix:



$$\boxed{\underset{6\times6}{[k]_e} = A_e t_e \underset{6\times3}{[B]^T}\underset{3\times3}{[D]}\underset{3\times6}{[B]}}$$

Potential energy of loading in the CST element

potential energy of loading in a finite element:



$$W_e = \int\limits_{\Omega_e} \underset{1\times 2}{\lfloor X \rfloor} \underset{2\times 1}{\{u\}} \, d\Omega_e + \int\limits_{\Gamma_{pe}} \underset{1\times 2}{\lfloor p \rfloor} \underset{2\times 1}{\{u\}} \, d\Gamma_{pe} =$$

$$\underset{2\times 1}{\{u\}} = \underset{2\times 6}{[N]} \underset{6\times 1}{\{q\}_e}$$

$$= \int\limits_{\Omega_e} \underset{1\times 2}{\lfloor X \rfloor} \underset{2\times 6}{[N]} \underset{6\times 1}{\{q\}_e} \, d\Omega_e + \int\limits_{\Gamma_{pe}} \underset{1\times 2}{\lfloor p \rfloor} \underset{2\times 6}{[N]} \underset{6\times 1}{\{q\}_e} \, d\Gamma_{pe} =$$

$$= \left(\int\limits_{\Omega_e} \underset{1\times 2}{\lfloor X \rfloor} \underset{2\times 6}{[N]} \, d\Omega_e + \int\limits_{\Gamma_{pe}} \underset{1\times 2}{\lfloor p \rfloor} \underset{2\times 6}{[N]} \, d\Gamma_{pe}\right) \underset{6\times 1}{\{q\}_e} = \left(\underset{1\times 6}{\lfloor F^X \rfloor_e} + \underset{1\times 6}{\lfloor F^p \rfloor_e}\right) \underset{6\times 1}{\{q\}_e} = \underset{1\times 6}{\lfloor F \rfloor_e} \underset{6\times 1}{\{q\}_e}$$

$$\underset{1\times 6}{\lfloor F^X \rfloor_e} = t_e \int\limits_{A_e} \underset{1\times 2}{\lfloor X \rfloor} \underset{2\times 6}{[N]} dA_e \quad ; \quad \underset{1\times 6}{\lfloor F^p \rfloor_e} = t_e \int_0^l \underset{1\times 2}{\lfloor p \rfloor} \underset{2\times 6}{[N]} \, ds$$

Components of equivalent load vector in the CST element

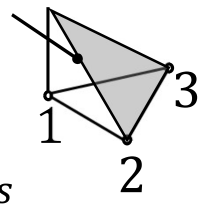
equivalent load vector due to mass forces:

$$\underset{1\times 6}{\lfloor F^X \rfloor_e} = t_e \int_{A_e} \lfloor X, Y \rfloor \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} dA_e =$$

$$= t_e \int_{A_e} \lfloor XN_1, YN_1, XN_2, YN_2, XN_3, YN_3 \rfloor dA_e = \left\lfloor F^X_{1e}, F^X_{2e}, F^X_{3e}, F^X_{4e}, F^X_{5e}, F^X_{6e} \right\rfloor$$

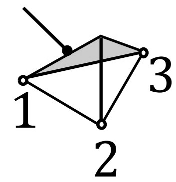
equivalent load vector due to surface load:

$$N_1(s)|_{1-2} = 1 - \frac{s}{l}$$



$$\underset{1\times 6}{\lfloor F^p \rfloor_e} = t_e \int_0^l \left\lfloor p_x, p_y \right\rfloor \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} ds =$$

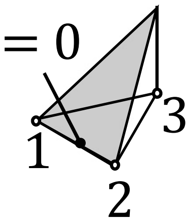
$$N_2(s)|_{1-2} = \frac{s}{l}$$



$$= t_e \int_0^l \left\lfloor p_x, p_y \right\rfloor \begin{bmatrix} 1-\frac{s}{l} & 0 & \frac{s}{l} & 0 & 0 & 0 \\ 0 & 1-\frac{s}{l} & 0 & \frac{s}{l} & 0 & 0 \end{bmatrix} ds =$$

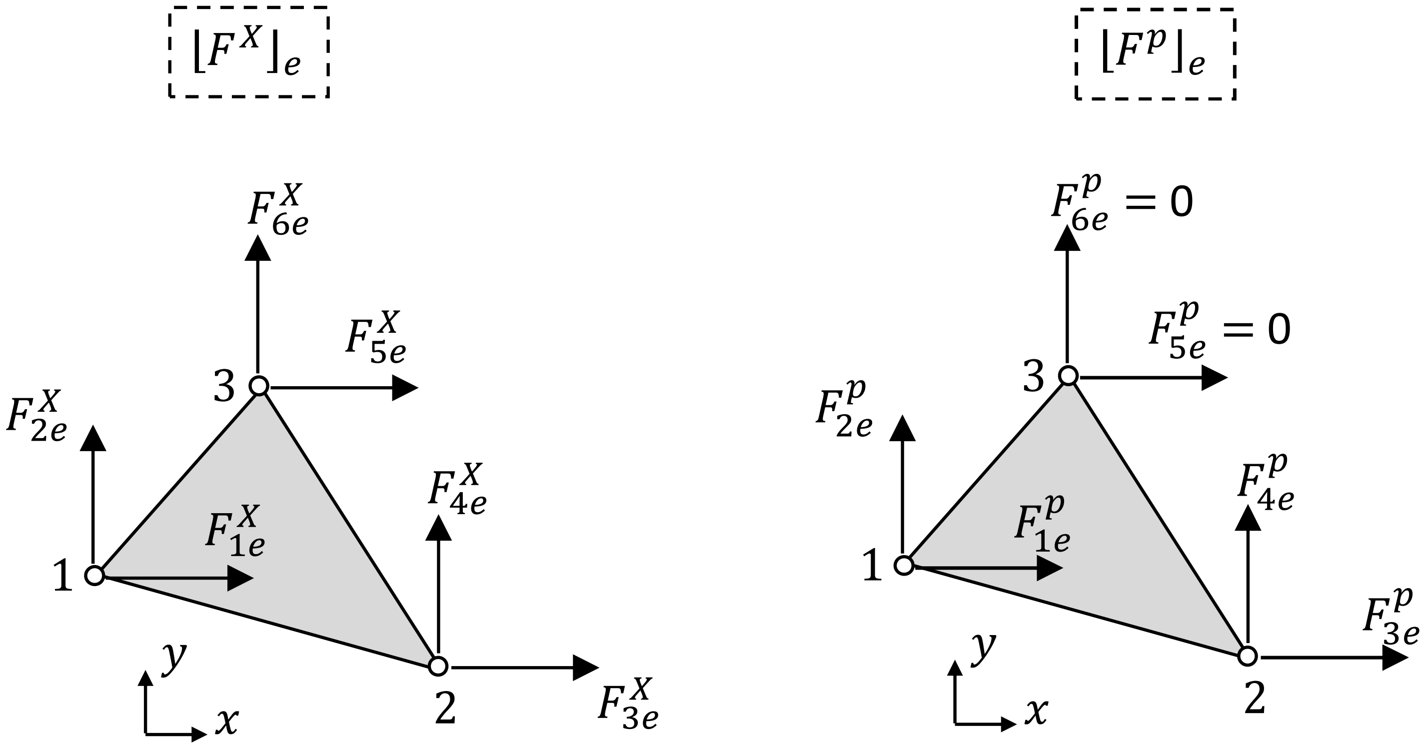
$$= t_e \int_0^l \left\lfloor p_x(1-\tfrac{s}{l}), p_y(1-\tfrac{s}{l}), p_x \tfrac{s}{l}, p_y \tfrac{s}{l}, 0,0 \right\rfloor ds =$$

$$N_3(x,y)|_{1-2} = 0$$



$$= \left\lfloor F^p_{1e}, F^p_{2e}, F^p_{3e}, F^p_{4e}, F^p_{5e}, F^p_{6e} \right\rfloor$$

Equivalent load vector in the CST element

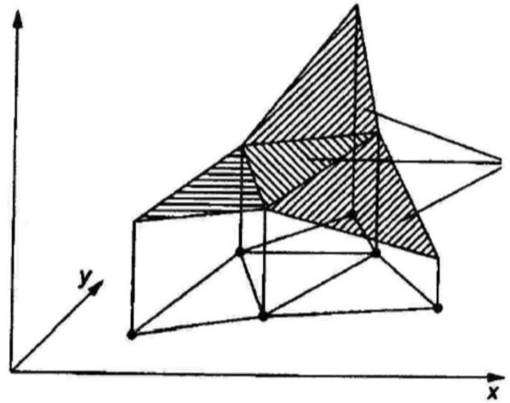


equivalent load vector:

$$\lfloor F \rfloor_e = \left\lfloor F^X_{1e} + F^p_{1e}, F^X_{2e} + F^p_{2e}, F^X_{3e} + F^p_{3e}, F^X_{4e} + F^p_{4e}, F^X_{5e} + F^p_{5e}, F^X_{6e} + F^p_{6e} \right\rfloor$$

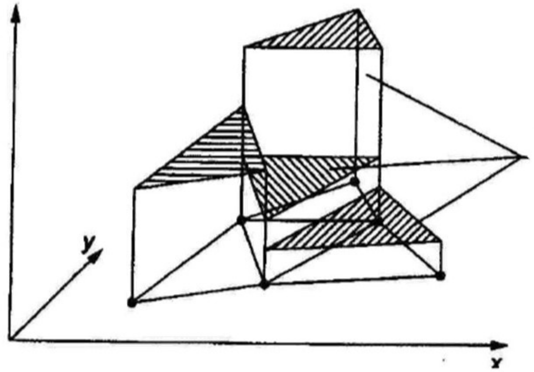
Results in the CST element

DOF solution : $u(\mathrm{x}, \mathrm{y}), v(\mathrm{x}, \mathrm{y})$



linear functions of coordinates (x, y)

element solution: $\underset{3\times 1}{\{\sigma\}}, \underset{3\times 1}{\{\varepsilon\}}$



constant